# On the initial value problem of a periodic box-ball system

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**Abstract**

We show that the initial value problem of a periodic box-ball system can be solved in an elementary way using simple combinatorial methods.

A periodic box-ball system (PBBS) is a dynamical system of balls in an array of boxes with a periodic boundary condition [1, 2]. The PBBS is obtained from the discrete KdV equation and the discrete Toda equation, both of which are known as typical integrable nonlinear discrete equations, through a limiting procedure called ultradiscretization [3, 4]. Since the ultradiscretization preserves the main properties of the original discrete equations, and the solvability of the initial value problem being an important property of integrable equations, we expect that the initial value problem of the PBBS can also be solved. In fact, the initial value problem for the PBBS was first solved by inverse ultradiscretization combined with the method of inverse scattering transform of the discrete Toda equation [5] and recently by the Bethe ansatz for an integrable lattice model with quantum group symmetry at the deformation parameter $q = 0$ and $q = 1$ [6]. These two methods, however, require fairly specialized mathematical knowledge on algebraic curves or representation theory of quantum algebras.

An important property which characterizes a state of the PBBS is the fundamental cycle of the state, *i.e.*, the length of the trajectory to which it belongs. Its explicit formula as well as statistical distribution was obtained and its relation to the celebrated Riemann hypothesis was clarified [7, 8, 9]. To prove the formula for fundamental cycle, one of the key steps is to compare a state with its 'reduced states' constructed by the '10-elimination'. In this article, we show that the initial value problem of the PBBS is solved by simple combinatorial arguments – essentially given in Ref. [7] – with some remarkable features of the reduced states.

First we quickly review the definition of the PBBS and its conserved quantities. Consider a one-dimensional array of boxes each with a capacity of one ball. A periodic boundary condition is imposed by assuming that the last box is adjacent to the first one. Let the number of boxes be $N$ and that of balls be $M$. We assume $M < N/2$. An arrangement of $M$ balls in $N$ boxes is called a state of the PBBS. Denoting a vacant box by 0 and a filled box by 1, a state

of the PBBS is represented by a $0,1$ sequence of length $N$. The time evolution rule from time step $t$ to $t+1$ can be described as follows:

- For a given state, connect all 10 pairs in the sequence with arc lines. We call them '$1^{\cap}$arc lines'.
- Neglecting the 10 pairs which are connected in the first step, connect all the remaining 10s with arc lines. We call them '$2^{\cap}$arc lines'.
- Repeat the above procedure until all 1s are connected to 0s with arc lines.
- Exchange all the 1s and 0s which are connected with arc lines. Then we obtain a new sequence which we call the state evolved by one time step.

If we denote by $p_j(t)$ the number of $j^{\cap}$arc lines, we obtain a nonincreasing sequence of positive integers, $p_j(t)$ $(j=1,2,3,\ldots,m)$. Then, this sequence is conserved in time, that is,

$$p_j(t) = p_j(t+1) \equiv p_j \qquad (j=1,2,3,\ldots,m).$$

As the sequence $(p_1, p_2, \ldots, p_m)$ is nonincreasing, we can associate a Young diagram to it by regarding $p_j$ as the number of squares in the $j$th column of the diagram. The lengths of the rows are also weakly decreasing positive integers. Let the distinct row lengths be $L_1 > L_2 > \cdots > L_s$ and let $n_j$ be the number of times that length $L_j$ appears. The set $\{L_j, n_j\}_{j=1}^{s}$ is another expression for the conserved quantities of the PBBS.

For example, for a state of the PBBS with $N=32,\ M=14$

$$(\sharp) \qquad 00111011100100011110001101000000,$$

the arc lines are drawn as in Fig. 1 and its conserved quantities are expressed by the Young diagram given in Fig. 2.

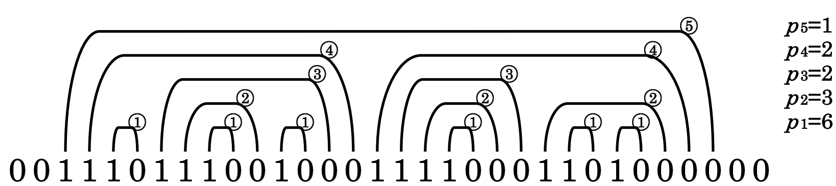


Figure 1: Arc lines and conserved quantities for the state $(\sharp)$.

To solve the initial value problem of the PBBS, we utilize the fact that a state of the PBBS is determined by its conserved quantities and the positions of 10 pairs to which the '10-elimination' is applied. In the present context, a 10-elimination is to convert a state to a state with smaller number of entries by eliminating all 10 pairs in the sequence connected with $1^{\cap}$arc lines. Let us consider a state $\mathcal{S}$ with conserved quantities $(p_1, p_2, \ldots, p_m)$ or equivalently $\{L_j, n_j\}_{j=1}^{s}$. Note that $s \le m$ and $L_1 = m$. We define its $k$-reduced state $(k=1,2,\ldots,m)$ as the one obtained from the state by eliminating all the 10 pairs connected with $j^{\cap}$arc lines for all $j:\ 1 \le j \le k$. The length of the $k$-reduced state is $N - 2\sum_{j=1}^{k} p_j$. We sometimes call the original state $\mathcal{S}$ as

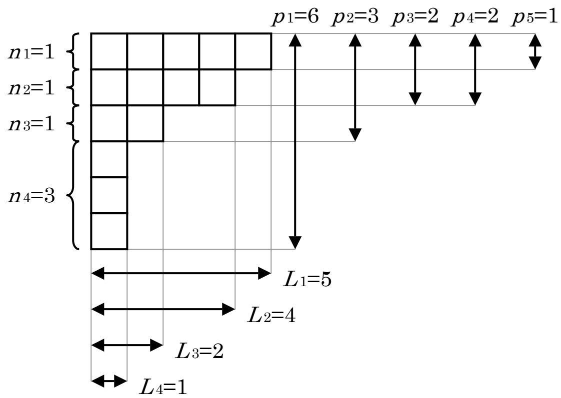


Figure 2: Young diagram corresponding to the conserved quantities of ($\sharp$).

0-reduced state. Apparently 10-elimination is not a reversible operation. If, however, we remember the places where 10 pairs have been eliminated, we can recover the original state, at least up to shift, by inserting 10 pairs there. Since there has necessarily been a 10 pair between consecutive 1 and 0 in the reduced state, we have only to remember the places for the other 10 pairs. (Such places are called the positions of 0-solitons in Ref. [7], and we use the same terminology here.) To specify the positions for insertion of 10 pairs, we number a place between $j$ and $j+1$th consecutive entries in a state by integer $j$. Note that, due to the periodic boundary condition, we set $0 \equiv N - 2\sum_{j=1}^{k} p_j$ for the $k$-reduced state. Hereafter we explicitly write the $j$th place by ‘ $\overset{j}{|}$ ’ in the $0, 1$ sequence. For example, the state ($\sharp$) is expressed as

$$\begin{array}{l} {\scriptstyle 0\ \ 1\ \ 2\ \ 3\ \ 4\ \ 5\ \ 6\ \ 7\ \ 8\ \ 9\ \ 10\ 11\ 12\ 13\ 14\ 15\ 16\ 17\ 18\ 19\ 20\ 21\ 22\ 23\ 24\ 25\ 26\ 27\ 28\ 29\ 30\ 31\ 32} \\ |0|0|1|1|1|0|1|1|1|0|0|1|0|0|0|1|1|1|1|0|0|0|1|1|0|1|0|0|0|0|0|0| \end{array} .$$

If we denote by $\hat{E}$ a 10-elimination, for the above state ($\sharp$), the $k$-reduced state is expressed as $\hat{E}^k(\sharp)$ and we have

$$\begin{aligned}
\hat{E}(\sharp) &= \begin{array}{l} {\scriptstyle 0\ \ 1\ \ 2\ \ 3\ \ 4\ \ 5\ \ 6\ \ 7\ \ 8\ \ 9\ \ 10\ 11\ 12\ 13\ 14\ 15\ 16\ 17\ 18\ 19\ 20} \\ |0|0|1|1|1|1|0|0|0|1|1|1|0|0|1|0|0|0|0|0| \end{array}, \\
\hat{E}^2(\sharp) &= \begin{array}{l} {\scriptstyle 0\ \ 1\ \ 2\ \ 3\ \ 4\ \ 5\ \ 6\ \ 7\ \ 8\ \ 9\ \ 10\ 11\ 12\ 13\ 14} \\ |0|0|1|1|1|0|0|1|1|0|0|0|0|0| \end{array}, \\
\hat{E}^3(\sharp) &= \begin{array}{l} {\scriptstyle 0\ \ 1\ \ 2\ \ 3\ \ 4\ \ 5\ \ 6\ \ 7\ \ 8\ \ 9\ \ 10} \\ |0|0|1|1|0|1|0|0|0|0| \end{array}, \\
\hat{E}^4(\sharp) &= \begin{array}{l} {\scriptstyle 0\ \ 1\ \ 2\ \ 3\ \ 4\ \ 5\ \ 6} \\ |0|0|1|0|0|0| \end{array}, \\
\hat{E}^5(\sharp) &= \begin{array}{l} {\scriptstyle 0\ \ 1\ \ 2\ \ 3\ \ 4} \\ |0|0|0|0| \end{array} .
\end{aligned}$$

In $\hat{E}(\sharp)$ the positions of 0-soliton are 4, 7 and 15, in $\hat{E}^2(\sharp)$ the position of 0-soliton is 10, in $\hat{E}^3(\sharp)$ there are no 0-solitons, and so on. Note that there is indeterminacy in constructing the original state from the reduced sequence; for example, both 0110001 and 1100010 turn to the same reduced state with the same positions of 0-solitons by 10-elimination. Hence we cannot necessarily

determine the exact position of the 10 pair in the original state from the position of corresponding 0-solitons in the reduced state.

In the $k$-reduced states, $\hat{E}^k(\mathcal{S})$, the number of positions of 0-solitons is $p_k - p_{k+1}$ $(p_{m+1} := 0)$. Hence there appear $\displaystyle\sum_{j=1}^{s} n_j = p_1$ 0-solitons in total in the reduced states. We also find that 0-solitons appear only at the $L_j$-reduced states $\hat{E}^{L_j}(\mathcal{S})$ $(j = 1, 2, \ldots, s)$ and that the number of 0-solitons is $n_j$ in $\hat{E}^{L_j}(\mathcal{S})$. There is no 0-soliton in the other reduced states. So we denote by $x_j^{(k)}$ $(k = 1, 2, \ldots, n_j)$ the position of the $k$th 0-soliton in the $L_j$-reduced state. Since $\hat{E}^{L_1}(\mathcal{S})$ only consists of $N - 2M$ 0s, we can reconstruct the original state up to some shift from $\{x_k^{(j)}\}_{j=1,k=1}^{s,\quad n_j}$. If, however, we know the original position in the 0-reduced state of one of the 0-solitons in the $L_1$-reduced state, we can recover the original state, because we have only to shift the state obtained by successive insertion of 10 pairs so that it coincides the original position.

For this purpose, it is more convenient to introduce a set of variables $\{\alpha_k^{(j)}\}_{j=1,k=1}^{s,\quad n_j}$ and $X_{s+1}$. First we define the reference positions $X_j$ $(j = 1, 2, \ldots, s+1)$ which is defined from $x_1^{(1)}$ recursively as follows.

- $\tilde{X}_{L_1} = x_1^{(1)}$.
- We denote by $\tilde{X}_{L_1-1}$ the position inbetween 1 and 0 in the $(L_1 - 1)$-reduced state of the 10 pair which turns into the 0-soliton at $\tilde{X}_{L_1}$ in the $L_1$-reduced state.
- Similarly we denote by $\tilde{X}_{L_1-2}$ the position inbetween 1 and 0 in the $(L_1 - 2)$-reduced state of the 10 pair which turns into the position $\tilde{X}_{L_1-1}$ in the $(L_1 - 1)$-reduced state. If there are more than one 10 pairs inserted at the position, $\tilde{X}_{L_1-2}$ is the left most position among them.
- Repeat the above procedure and obtain $\tilde{X}_k$ $(k = 0, 1, \ldots, L_1)$.
- $X_j := \tilde{X}_{L_j} \quad (j = 1, 2, \ldots, s+1,\ \text{where } L_{s+1} := 0)$.

In the above example (♯), $\tilde{X}_{L_1} = \tilde{X}_5 = 2,\ \tilde{X}_4 = 3,\ \tilde{X}_3 = 4,\ \tilde{X}_2 = 5,\ \tilde{X}_1 = 6$ and $\tilde{X}_0 = 9$. Hence $X_1 = 2,\ X_2 = 3,\ X_3 = 5,\ X_4 = 6$ and $X_5 = 9$. In the terminology of Ref. [7], $X_j$ is the position of one of the 'largest soliton' in the $L_j$-reduce state which turns to a 0-soliton in the $L_1$-reduced state. Hence $X_{s+1}$ is 'the original position in the 0-reduced state' of the 0-soliton at the position $x_1^{(1)}$.

Then we define $\alpha_k^{(j)}$ $(1 \le \alpha_k^{(j)} \le N_j)$ by

$$\alpha_k^{(j)} = X_j - x_k^{(j)} \mod N_j \quad (j = 1, 2, \ldots, s,\ \ k = 1, 2, \ldots, n_j), \tag{1}$$

where

$$N_j := N - 2M + \sum_{i=1}^{j} 2n_i(L_i - L_j) \qquad (j = 1, 2, \ldots, s)$$

is the number of entries in the $L_j$-reduced state. Note that $\alpha_1^{(1)} = N_1$ and $\alpha_k^{(j)}$ is the distance of the $k$th 0-soliton from the position of the 'largest soliton' in the $L_j$-reduce state, which turns to a 0-soliton in the $L_1$-reduced state. Since

the state $\mathcal{S}$ can be determined up to shift by inserting 10 pairs at the positions of 0-solitons and that between consecutive 1 and 0, and $X_{s+1}$ determines the amount of the shift, the state $\mathcal{S}$ is uniquely determined by the variables $\{\alpha_k^{(j)}\}_{j=1,\,k=1+\delta_{1,j}}^{s,\quad n_j}$ and $X_{s+1}$. Formally we may write that

$$\{\alpha_k^{(j)}\}_{k=1}^{n_j} \in S^{n_j}(\mathbb{Z}_{N_j}) := \underbrace{\mathbb{Z}_{N_j} \times \mathbb{Z}_{N_j} \times \cdots \times \mathbb{Z}_{N_j}}_{n_j} / S^{n_j} \quad (j = 2, 3, \ldots, s),$$

where $\mathbb{Z}_{N_j}$ is the cyclic group of order $N_j$, $S^{n_j}$ is the symmetric group of order $n_j$, and, since there are $N_1 + 1$ distinct positions[1] for $(n_1 - 1)$ 0-solitons in the $L_1$-reduced state,

$$\{\alpha_k^{(1)}\}_{k=2}^{n_1} \in S^{n_1-1}(\mathbb{Z}_{N_1+1}),$$

and $X_{s+1} \in \mathbb{Z}_N$. If we define

$$\tilde{V}_Y := S^{n_1-1}(\mathbb{Z}_{N_1+1}) \times S^{n_2}(\mathbb{Z}_{N_2}) \times \cdots \times S^{n_s}(\mathbb{Z}_{N_s}) \times \mathbb{Z}_N,$$

then an element of $\tilde{V}_Y$ naturally corresponds to a state of the PBBS. Since there are $n_1$ choices of the reference position, there are exactly $n_1$ elements in $\tilde{V}_Y$ which correspond to a state of the PBBS. We regard that one element in $\tilde{V}_Y$ is equivalent with another if and only if they corresponds to the same state of the PBBS. Denoting by $V_Y$ the quotient set of $\tilde{V}_Y$ according to this equivalence relation, we obtain the following theorem;

**Theorem 1**
*Denote by $\Omega_Y$ a set of the states of the PBBS of $M$ balls and $N$ boxes with conserved quantities $\{L_j,\ n_j\}_{j=1}^{s}$ characterized by the Young diagram $Y$. Then there is a one to one correspondence between an element of $\Omega_Y$ and that of $V_Y$, hence $\Omega_Y \cong V_Y$. The explicit bijection is given by the 10-eliminations and its inverse operations with shift using the variables $X_{s+1}$ and $\{\alpha_k^{(j)}\}_{j=1,\,k=1+\delta_{1,j}}^{s,\quad n_j}$.*

As for the explicit construction of a state of the PBBS from an element in $V_Y$, see the example given below.

Now we consider the initial value problem of the PBBS. Let $\mathcal{S}(t)$ be the state evolved from the state $\mathcal{S}$ by $t$ time steps. From Theorem 1, we have that the dynamics of the PBBS can be described by an element in $V_Y$. Hence, to determine a state $\mathcal{S}(t)$, it is enough to obtain the variables $\left(\{\alpha_k^{(j)}(t)\}_{j=1,\,k=1+\delta_{1,j}}^{s,\quad n_j},\ X_{s+1}(t)\right) \in V_Y$ from the initial values $\left(\{\alpha_k^{(j)}(0)\}_{j=1,\,k=1+\delta_{1,j}}^{s,\quad n_j},\ X_{s+1}(0)\right) \in V_Y$. However, the time dependence of these variables has already been given in Ref. [7]:

**Proposition 1 ([7] Theorem 3.1, Lemma 4.2 and Lemma 4.3)**
*For $i = 1, 2, \ldots, s+1$, let $\gamma_i(t)$ be $\gamma_1(t) := 0,\ \gamma_2(t) := (L_1 - L_2)t$ and*

$$\gamma_i(t) := (L_1 - L_i)t + 2\sum_{j=2}^{i-1}(L_j - L_i)\sum_{k=1}^{n_j}\beta_k^{(j)}(t)$$

[1] When another 0-soliton is located at position $X_1$, it is either at the left of the reference 0-soliton or to the right of it.

*for $i = 3, 4, \ldots, s+1$, where*

$$\beta_k^{(j)}(t) := \left\lfloor \frac{\gamma_j(t) + \alpha_k^{(j)}(0) - 1}{N_j} \right\rfloor \qquad (j = 2, 3, \ldots, s, \;\; k = 1, 2, \ldots, n_j).$$

*Then it holds that*

$$\alpha_k^{(j)}(t) = \alpha_k^{(j)}(0) + \gamma_j(t) \qquad \mathrm{mod}\ N_j.$$

*and*

$$X_{s+1}(t) = X_{s+1}(0) + \gamma_{s+1}(t) \qquad \mathrm{mod}\ N.$$

**Remark 1**
*Note that both $\gamma_i(t)$s and $\alpha_k^{(j)}(t)$s are determined recursively. As we shall see, practically we have only to use the relation*

$$\alpha_k^{(j)}(0) + \gamma_j(t) = N_j \beta_k^{(j)}(t) + \alpha_k^{(j)}(t).$$

In conclusion, we have solved the initial value problem of the PBBS which may be stated as

**Theorem 2**
*The initial value problem of the PBBS is solved in the space of $V_Y$. Its dynamics is explicitly given in Proposition 1.*

In the rest of this article, we explain how to obtain the time evolution of a state of the PBBS by means of an example. Suppose that we have the state ($\sharp$) which we have used previously

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at time step $t = 0$. From Fig. 2 and $\hat{E}^k(\sharp)$, we find $s = 4,\ N_1 = 4,\ N_2 = 6,\ N_3 = 14,\ N_4 = 20$. Then referring to the reduced sequences $\hat{E}^k(\sharp)$, we obtain

$$x_1^{(1)}(0) = 2;\ \ x_1^{(2)}(0) = 3;\ \ x_1^{(3)}(0) = 10;\ \ x_1^{(4)}(0) = 4,\ x_2^{(4)}(0) = 7,\ x_3^{(4)}(0) = 15.$$

The reference positions $X_j(0)$ $(j = 1, 2, \ldots, 5)$ are found to be

$$X_1(0) = 2,\ X_2(0) = 3,\ X_3(0) = 5,\ X_4(0) = 6,\ X_5(0) = 9.$$

Hence $\alpha_k^{(j)}(0)$s are given as

$$\alpha_1^{(1)}(0) = 4;\ \ \alpha_1^{(2)}(0) = 6;\ \ \alpha_1^{(3)}(0) = 9;\ \ \alpha_1^{(4)}(0) = 2,\ \alpha_2^{(4)}(0) = 19,\ \alpha_3^{(4)}(0) = 11.$$

Now let us consider the state at $t = 10000$. According to Proposition 1, $\alpha_k^{(j)}(t)$s

and $\gamma_i(t)$s are calculated recursively as follows;

(1) $N_2 = 6,\ \gamma_2(t) = (L_1 - L_2)t = 10000$

$$\Rightarrow\ \alpha_1^{(2)}(0) + \gamma_2(t) = 6 + 10000 = 1667 \cdot 6 + 4$$

$$\Rightarrow\ \alpha_1^{(2)}(t) = 4;$$

(2) $N_3 = 14,\ \gamma_3(t) = (L_1 - L_3) \cdot t + 2(L_2 - L_3) \cdot 1667 = 36668$

$$\Rightarrow\ \alpha_1^{(3)}(0) + \gamma_3(t) = 9 + 36668 = 2619 \cdot 14 + 11$$

$$\Rightarrow\ \alpha_1^{(3)}(t) = 11;$$

(3) $N_4 = 20,$

$$\gamma_4(t) = (L_1 - L_4) \cdot t + 2(L_2 - L_4) \cdot 1667 + 2(L_3 - L_4) \cdot 2619 = 55240$$

$$\Rightarrow \begin{cases} \alpha_1^{(4)}(0) + \gamma_4(t) = 2 + 55240 = 2762 \cdot 20 + 2 \\ \alpha_2^{(4)}(0) + \gamma_4(t) = 19 + 55240 = 2762 \cdot 20 + 19 \\ \alpha_3^{(4)}(0) + \gamma_4(t) = 11 + 55240 = 2762 \cdot 20 + 11 \end{cases}$$

$$\Rightarrow \begin{cases} \alpha_1^{(4)}(t) = 2, \\ \alpha_2^{(4)}(t) = 19, \\ \alpha_3^{(4)}(t) = 11; \end{cases}$$

(4) $N = 32,$

$$\begin{aligned} \gamma_5(t) = {} & (L_1 - L_5) \cdot t + 2(L_2 - L_5) \cdot 1667 + 2(L_3 - L_5) \cdot 2619 \\ & + 2(L_4 - L_5) \cdot (2762 + 2762 + 2762) = 90384 \end{aligned}$$

$$\begin{aligned} \Rightarrow\ X_5(t) &= X_5(0) + \gamma_5(t) \mod N = 9 + 90384 \mod 32 \\ &= 25. \end{aligned}$$

From these data, the state at $t = 10000$ is constructed up to shift by inserting 10 pairs as;

$$\begin{array}{ll}
 & \scriptstyle 0\ \ 1\ \ 2\ \ 3^o\ \ 4 \\
 & |0|0|0|0|, \\
\longrightarrow & \scriptstyle 0^o\ \ 1\ \ 2\ \ 3\ \ 4^*\ \ 5\ \ 6 \\
 & |0|0|0|1|0|0|, \\
\longrightarrow & \scriptstyle 0\ \ 1\ \ 2\ \ 3\ \ 4\ \ 5\ \ 6\ \ 7^*\ \ 8\ \ 9\ \ 10 \\
 & |1|0|0|0|0|1|1|0|0|0|, \\
\longrightarrow & \scriptstyle 0\ \ 1\ \ 2\ \ 3\ \ 4\ \ 5\ \ 6\ \ 7\ \ 8\ \ 9\ \ 10^*\ \ 11\ \ 12\ \ 13^o\ \ 14 \\
 & |1|1|0|0|0|0|0|1|1|1|0|0|0|0|, \\
\longrightarrow & \scriptstyle 0\ \ 1\ \ 2^o\ \ 3\ \ 4\ \ 5\ \ 6\ \ 7\ \ 8\ \ 9\ \ 10\ \ 11^o\ \ 12\ \ 13^*\ \ 14^o\ \ 15\ \ 16\ \ 17\ \ 18\ \ 19\ \ 20 \\
 & |1|1|1|0|0|0|0|0|0|1|1|1|1|0|0|0|0|1|0|0|, \\
\longrightarrow & \scriptstyle 0\ \ 1\ \ 2\ \ 3\ \ 4\ \ 5\ \ 6\ \ 7\ \ 8\ \ 9\ \ 10\ \ 11\ \ 12\ \ 13\ \ 14\ \ 15\ \ 16\ \ 17\ \ 18\ \ 19\ \ 20^*\ \ 21\ \ 22\ \ 23\ \ 24\ \ 25\ \ 26\ \ 27\ \ 28\ \ 29\ \ 30\ \ 31\ \ 32 \\
 & |1|1|1|0|1|1|0|0|0|0|0|0|0|1|1|1|0|1|1|1|0|0|1|0|0|0|0|1|1|0|0|0|\ .
\end{array}$$

Here ' $\overset{j^*}{|}$ ' denotes the position of the largest soliton (referring position) and ' $\overset{j^o}{|}$ ' denotes that of a 0-soliton. The position $x_1^{(1)}(t)$ can be chosen arbitrary, and we took $x_1^{(1)}(t) = 3$ in this example. Finally we translate the above state so that the position of the largest soliton coincides with $X_{s+1}(t)$ as

$$\begin{array}{l}
\scriptstyle 0\ \ 1\ \ 2\ \ 3\ \ 4\ \ 5\ \ 6\ \ 7\ \ 8\ \ 9\ \ 10\ \ 11\ \ 12\ \ 13\ \ 14\ \ 15\ \ 16\ \ 17\ \ 18\ \ 19\ \ 20\ \ 21\ \ 22\ \ 23\ \ 34\ \ 25^*\ \ 26\ \ 27\ \ 28\ \ 29\ \ 30\ \ 31\ \ 32 \\
|1|1|0|0|0|1|1|1|0|1|1|0|0|0|0|0|0|0|1|1|1|0|1|1|1|0|0|1|0|0|0|0|\ .
\end{array}$$

This is the state at $t = 10000$.

In this article, we have solved the initial value problem in an elementary way. We also remark that our method can be equally applied to extended PBBSs with carrier capacity $\ell$ as those treated in Ref. [12]. In these PBBSs, we have only to replace $L_j \to \min[L_j, \ell]$ in Proposition 1 and apply the above procedures. Clarifying the relation between our methods and previous work based on algebraic curves and representation theories, is one of the important problems we want to address in the future.

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